Heuristic Proof of Itō's Lemma

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The general Itō Process is given by the SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t \tag{1}$$

Define the function $f(X_t, t)$. Then the 2nd order Taylor series expansion for f is

$$\begin{aligned} & f(X_{t+dt}, t+dt) - f(X_t, t) \\ = & \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial X_t}(dX_t) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t \partial X_t}(dX_t dt). \end{aligned}$$

Hence

$$df(X_t, t) = \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial X_t}(dX_t) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t \partial X_t}(dX_t dt) \tag{2}$$

since $(dt)^2 = 0$. Now substitute dX_t from equation (1) into equation (2) and use the fact that $dW_t dt = 0$ and $(dW_t)^2 = dt$.

$$\begin{aligned} df(X_t, t) &= \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial X_t}(\mu dt + \sigma dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(\mu dt + \sigma dW_t)^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial t \partial X_t}(\mu dt + \sigma dW_t)dt \\ &= \frac{\partial f}{\partial t}(dt) + \frac{\partial f}{\partial X_t}(\mu dt + \sigma dW_t) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\left(\sigma^2 dt\right). \end{aligned}$$

Rearrange terms to get Itō's Lemma

$$df(X_t, t) = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X_t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X_t^2}\right)dt + \left(\sigma\frac{\partial f}{\partial X_t}\right)dW_t$$

where $\mu = \mu(X_t, t)$ and $\sigma = \sigma(X_t, t)$ is written for notational convenience.